Journal of Sound and Vibration (1998) **218**(5), 769–789
Article No. sv981830

STOCHASTIC STABILITY OF QUASI-NON-INTEGRABLE-HAMILTONIAN SYSTEMS

W. Q. Zhu and Z. L. Huang
Department of Mechanics, Zhejiang University, Hangzhou 310027, *People's Republic of China*

(*Received* 17 *October* 1997, *and in final form* 17 *June* 1998)

An n-degree-of-freedom quasi-non-integrable-Hamiltonian system is first reduced to an *Itô* equation of one-dimensional averaged Hamiltonian by using the stochastic averaging method developed by the first author and his coworkers. The necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution of the quasi-non-integrable-Hamiltonian system are then obtained approximately by examining the sample behaviors of the one-dimensional diffusion process of the square-root of averaged Hamiltonian at the two boundaries. A system of linearly and non-linearly coupled two non-linearly damped oscillators subject to parametric excitations of Gaussian white noises is employed as an example to illustrate the procedure, and the effects of non-linear damping and non-linear coupling on the stability are analyzed in detail.

1. INTRODUCTION

Motion stability is a basic qualitative behavior of a dynamical system and it is defined in terms of boundedness near and convergence to a known solution. Instability of a dynamic system may lead to failure of the system. Stability or instability is thus a very important topic for the dynamical systems in science and engineering.

There are many types of stability definitions for deterministic dynamical systems and even more for stochastic dynamical systems [1], since convergence of a stochastic sequence can be interpreted in several different ways. Stochastic stability has been investigated extensively for linear systems. For non-linear stochastic systems, very few results have been obtained [2]. For a dynamical system governed by a one-dimensional *Itô* stochastic differential equation, Kozin and Sunahara [3], Sri Namachchivaya [4] and Zhang [5] showed that the necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution can be obtained by examining the sample behavior of the system at the trivial solution. Recently, Lin and Cai [6] showed that the sample behaviors at both of the two boundaries, not just the one at the trivial solution, should be examined in order that the necessary and sufficient conditions for the asymptotic stability in probability can be obtained. They concluded that the trivial solution was

0022–460X/98/500769 + 22 $30.00/0

asymptotic stable if and only if it was a exit or attractively natural, and the other boundary was an entrance or repulsively natural. They reduced a non-linear damped oscillator and an oscillator with non-linear restoring force subject to parametric excitation of Gaussian white noise to a one-dimensional diffusion process, using stochastic averaging [7] and quasi-conservative averaging [8, 9], respectively. Then they obtained the necessary and sufficient conditions for the asymptotic stability in probability of the systems from examining the sample behaviors of the averaged one-dimensional diffusion process at the two boundaries.

It has been shown that the response of a quasi-non-integrable-Hamiltonian system (a non-integrable-Hamiltonian system of n-degrees-of-freedom (DOF) subject to lightly linear and/or non-linear damping and parametric and (or) external excitations of Gaussian white noises of small intensities) converges in probability to a one-dimensional diffusion process of averaged Hamiltonian as the damping and excitations approach zero, and therefore, the stochastic averaging method for quasi-non-integrable-Hamiltonian systems has been developed by the first author and his coworker [10]. Thus, it is possible in principle to determine approximately the necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution of a quasi-non-integrable-Hamiltonian system by examining the sample behaviors of the one-dimensional diffusion process of the square-root of the averaged Hamiltonian at the two boundaries. However, the drift and diffusion coefficients of the one-dimensional diffusion process of the square-root of averaged Hamiltonian are defined by quite complicated multi-fold integrals. Asymptotic estimation of the integrals is thus necessary to examine the sample behaviors of the one-dimensional diffusion process at the boundaries.

In the present paper, the criteria for the classification of the boundary of the one-dimensional diffusion process are first reviewed and then the stochastic averaging method for quasi-non-integrable-Hamiltonian systems is introduced. After that the asymptotic stability in probability in terms of the square-root of averaged Hamiltonian is defined and it is shown how the necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution of a quasi-non-integrable-Hamiltonian system can be obtained approximately by examining the sample behaviors of the one-dimensional diffusion process of the square-root of the averaged Hamiltonian.

2. BOUNDARY CLASSIFICATION

Consider a one-dimensional diffusive Markov process $X(t)$ defined on $[x_l, x_r]$ and governed by an *Itô* stochastic differential equation

$$\mathrm{d}X(t) = m(X)\,\mathrm{d}t + \sigma(X)\,\mathrm{d}B(t), \tag{1}$$

where $m(X)$ and $\sigma(X)$ are known as the drift and diffusion coefficients, respectively, and $B(t)$ is a unit Wiener process. The left boundary x_l may or may not be $-\infty$, and the right boundary x_r may or may not be $+\infty$. The behavior of a one-dimensional diffusion process at or near a boundary can be described by the following four functions [11, 12]

$$l(x) = \int_{x_0}^{x} \psi^{-1}(u)\,\mathrm{d}u, \quad v(x) = \int_{x_0}^{x} \frac{\psi(u)}{\sigma^2(u)}\,\mathrm{d}u, \qquad \Sigma(x) = \int_{x_0}^{x} v(u)\,\mathrm{d}l(u),$$

$$N(x) = \int_{x_0}^{x} l(u)\,\mathrm{d}v(u), \tag{2}$$

where

$$\psi(x) = \exp\left[\int \frac{2m(x)}{\sigma^2(x)}\,\mathrm{d}x\right]. \tag{3}$$

$l(x)$ is a scale function, $v(x)$ is a speed function, $\Sigma(x)$ is a measure of the time to reach a point x from an interior point x_0, and $N(x)$ is a measure of the time to reach an interior point x_0 from a point x. A boundary of one-dimensional diffusion process may be classified as follows. (1) Regular boundary: the process can either reach the boundary from an interior point, or reach an interior point from the boundary. (2) Exit boundary: the process can reach the boundary from an interior point, but cannot reach an interior point from the boundary. (3) Entrance boundary: the process can reach an interior point from the boundary, but cannot reach the boundary from an interior point. (4) Natural boundary: the process cannot reach the boundary from an interior point, nor can it reach an interior point from the boundary.

In the above statements, ‘‘reach’’ means accessibility in finite time. A natural boundary may be further classified as attractively natural, strictly natural and repulsively natural [6, 12]. These various types of boundaries can be identified according to the values of $l(x_b)$, $v(x_b)$, $\Sigma(x_b)$ and $N(x_b)$, as shown in Table 1 (Tables 1–4 are reproduced from references [2, 6]).

TABLE 1†

Criteria‡				Classifications	
$l(x_b)$	$v(x_b)$	$\Sigma(x_b)$	$N(x_b)$		
$<\infty$*	$<\infty$*	$<\infty$	$<\infty$	Regular	Accessible
$<\infty$*	$=\infty$*	$<\infty$*	$=\infty$	Exit	
$<\infty$*	$=\infty$*	$=\infty$*	$=\infty$	Attractively natural	
$=\infty$*	$<\infty$*	$=\infty$	$=\infty$	Repulsively natural	Inaccessible
$=\infty$*	$=\infty$*	$=\infty$	$=\infty$	Strictly natural	
$=\infty$*	$<\infty$	$=\infty$	$<\infty$*	Entrance	

† Modified from the original table of Karlin and Taylor [12].

‡ The asterisk indicate the minimal sufficient conditions for each type of boundary. For example, the minimal sufficient conditions for a regular boundary are $l(x_b) < \infty$ and $v(x_b) < \infty$.

TABLE 2

Classification of singular boundary of the first kind†

State	Conditions				Class
	$\alpha_s < 1$				Regular
$\sigma(x_s) = 0$	$\alpha_s = 1$	$m(x_l) < 0$			Exit
$(\alpha_s > 0)$		or $m(x_r) > 0$			
$m(x_s) \neq 0$		$m(x_l) > 0$	$0 < c_s < 1$		Regular
$(\beta_s = 0)$		or $m(x_r) < 0$	$c_s \geqslant 1$		Entrance
(shunt)	$\alpha_s > 1$	$m(x_l) < 0$			Exit
		or $m(x_r) > 0$			
		$m(x_l) > 0$			Entrance
		or $m(x_r) < 0$			
$\sigma(x_s) = 0$	$\alpha_s < 1 + \beta_s$	$\alpha_s < 1$			Regular
$(\alpha_s > 0)$		$1 \leqslant \alpha_s < 2$			Exit
$m(x_s) = 0$		$\alpha_s \geqslant 2$			Attractively
$(\beta_s > 0)$					natural
(trap)	$\alpha_s > 1 + \beta_s$	$\beta_s < 1$	$m(x_l^+) < 0$		Exit
			or $m(x_r^-) > 0$		
			$m(x_l^+) > 0$		Entrance
			or $m(x_r^-) < 0$		
		$\beta_s \geqslant 1$	$m(x_l^+) < 0$		Attractively
			or $m(x_r^-) > 0$		natural
			$m(x_l^+) > 0$		Repulsively
			or $m(x_r^-) < 0$		natural
	$\alpha_s = 1 + \beta_s$	$\beta_s < 1$	$c_s > \beta_s$	$c_s \geqslant 1$	Entrance
				$c_s < 1$	Regular
			$c_s \leqslant \beta_s$		Exit
		$\beta_s \geqslant 1$	$c_s > \beta_s$		Repulsively
					natural
			$c_s \leqslant \beta_s$	$c_s \geqslant 1$	Strictly
					natural
				$c_s < 1$	Attractively
					natural

A boundary may be singular. It is called a singular boundary of the first kind if the diffusion coefficient σ vanishes, and a singular boundary of the second kind if the drift coefficient m becomes unbounded. Further classification of a singular boundary is based on the limiting behaviors of the drift and diffusion coefficients near the boundary. Specially, singular boundary of the first kind is classified depending on the values of the following parameters (see Table 2):

diffusion exponent α_s

$$\sigma^2(x) = O|x - x_s|^{\alpha_s}, \quad \alpha_s \geqslant 0, \quad \text{as} \quad x \to x_s, \tag{4}$$

drift exponent β_s

$$m(x) = O|x - x_s|^{\beta_s}, \quad \beta_s \geqslant 0, \quad \text{as} \quad x \to x_s, \tag{5}$$

character value c_s

$$c_l = \lim_{x\to x_l^+} \frac{2m(x)(x-x_l)^{\alpha_l-\beta_l}}{\sigma^2(x)}, \qquad c_r = -\lim_{x\to x_r^-} \frac{2m(x)(x_r-x)^{\alpha_r-\beta_r}}{\sigma^2(x)}. \tag{6}$$

The classification of a singular boundary of the second kind is also identified based on the values of the diffusion exponent α_s, drift exponent β_s and the character value c_s (see Tables 3 and 4 in Appendix A), but these parameters are defined slightly differently as follows:

For $|x_s| < \infty$,

diffusion exponent α_s

$$\sigma^2(x) = O|x-x_s|^{-\alpha_s}, \quad \alpha_s \geqslant 0, \quad \text{as} \quad x\to x_s, \tag{7}$$

drift exponent β_s

$$m(x) = O|x-x_s|^{-\beta_s}, \quad \beta_s \geqslant 0, \quad \text{as} \quad x\to x_s, \tag{8}$$

character value c_s

$$c_l = \lim_{x\to x_l^+} \frac{2m(x)(x-x_l)^{\beta_l-\alpha_l}}{\sigma^2(x)}, \qquad c_r = -\lim_{x\to x_r^-} \frac{2m(x)(x_r-x)^{\beta_r-\alpha_r}}{\sigma^2(x)}. \tag{9}$$

For $|x_s| = \infty$,

diffusion exponent α_s

$$\sigma^2(x) = O|x|^{\alpha_s}, \quad \alpha_s \geqslant 0, \quad \text{as} \quad |x|\to\infty, \tag{10}$$

TABLE 3

Classification of singular boundary of the second kind ($|x_s| < \infty$)

State	Conditions			Class
$\lvert m(x_s)\rvert = \infty$	$\beta_s < 1$			Regular
$(\beta_s > 0)$	$\beta_s = 1$	$c_s \leqslant -1$		Exit
$\sigma(x_s) < \infty$		$-1 < c_s < 1$		Regular
$(\alpha_s = 0)$		$c_s \geqslant 1$		Entrance
	$\beta_s > 1$	$m(x_l^+) < 0$ or $m(x_r^-) > 0$		Exit
		$m(x_l^+) > 0$ or $m(x_r^-) < 0$		Entrance
$\lvert m(x_s)\rvert = \infty$	$\beta_s < 1 + \alpha_s$			Regular
$(\beta_s > 0)$	$\beta_s > 1 + \alpha_s$	$m(x_l^+) < 0$ or $m(x_r^-) > 0$		Exit
$\sigma(x_s) = \infty$ $(\alpha_s > 0)$		$m(x_l^+) > 0$ or $m(x_r^-) < 0$		Entrance
	$\beta_s = 1 + \alpha_s$	$c_s \geqslant -\beta_s$	$c_s \geqslant 1$	Entrance
			$c_s < 1$	Regular
		$c_s < -\beta_s$		Exit

TABLE 4

Classification of singular boundary of the second kind at infinity

State	Conditions				Class
$m(\infty)\| = \infty$	$m(-\infty) < 0$	$\beta_s > 1$			Exit
$(\beta_s > 0)$	or $m(+\infty) > 0$	$\beta_s \leqslant 1$			Attractively natural
$\sigma(\infty) < \infty$					
$(\alpha_s = 0)$	$m(-\infty) > 0$	$\beta_s > 1$			Entrance
	or $m(+\infty) < 0$	$\beta_s \leqslant 1$			Repulsively natural
$\|m(\infty)\| = \infty$	$\beta_s > \alpha_s - 1$	$m(-\infty) < 0$	$\beta_s > 1$		Exit
$(\beta_s > 0)$		or $m(+\infty) > 0$	$\beta_s \leqslant 1$		Attractively natural
$\sigma(\infty) = \infty$					
$(\alpha_s > 0)$		$m(-\infty) > 0$	$\beta_s > 1$		Entrance
		or $m(+\infty) < 0$	$\beta_s \leqslant 1$		Repulsively natural
	$\beta_s < \alpha_s - 1$				Regular
	$\beta_s = \alpha_s - 1$	$\beta \leqslant 1$	$c_s > -\beta_s$		Repulsively natural
			$c_s \leqslant -\beta_s$	$c_s \geqslant -1$	Strictly natural
				$c_s < -1$	Attractively natural
		$\beta_s > 1$	$c_s > -\beta_s$	$c_s \geqslant -1$	Entrance
				$c_s < -1$	Regular
			$c_s \leqslant -\beta_s$		Exit

drift exponent β_s

$$m(x) = O|x|^{\beta_s}, \quad \beta_s \geqslant 0, \quad \text{as} \quad |x| \to \infty, \tag{11}$$

character value c_s

$$c_l = \lim_{x \to -\infty} \frac{2m(x)|x|^{\alpha_l - \beta_l}}{\sigma^2(x)}, \qquad c_r = -\lim_{x \to \infty} \frac{2m(x)|x|^{\alpha_r - \beta_r}}{\sigma^2(x)}. \tag{12}$$

3. STOCHASTIC AVERAGING OF QUASI-NON-INTEGRABLE-HAMILTONIAN SYSTEMS

Consider an n DOF stochastically excited and dissipated Hamiltonian system governed by the following n pairs of equations of motion

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} \tag{13a}$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} - \varepsilon c_{ij} \frac{\partial H'}{\partial P_j} + \varepsilon^{1/2} f_{ik} w_k(t), \qquad i, j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m, \tag{13b}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H' = H'(\mathbf{Q}, \mathbf{P})$ is a Hamiltonian with continuous second-order derivatives;

$c_{ij} = c_{ij}(\mathbf{Q}, \mathbf{P})$ are differentiable functions; $f_{ik} = f_{ik}(\mathbf{Q}, \mathbf{P})$ are twice-differentiable functions; ε is a small parameter; and $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions

$$E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau). \tag{14}$$

The system governed by equations (13a) and (13b) is termed a quasi-Hamiltonian one and it is generally non-linear. The first summation terms on the right side of equation (13b) may represent a set of linear and (or) non-linear damping mechanisms, while the second summation terms may include external and (or) parametric excitations of Gaussian white noises.

Equations (13a) and (13b) are equivalent to the following set of *Itô* stochastic differential equations

$$\mathrm{d}Q_i = \frac{\partial H'}{\partial P_i}\,\mathrm{d}t, \tag{15a}$$

$$\mathrm{d}P_i = \left(-\frac{\partial H'}{\partial Q_i} - \varepsilon c_{ij}\frac{\partial H'}{\partial P_j} + \varepsilon D_{kl} f_{jl}\frac{\partial f_{ik}}{\partial P_j}\right)\mathrm{d}t + \varepsilon^{1/2} f_{ik}\,\mathrm{d}B_k(t),$$

$$i, j = 1, 2, \ldots, n; \quad k, l = 1, 2, \ldots, m, \tag{15b}$$

where $B_k(t)$ are the Wiener processes. The double summation terms on the right side of equation (15b) are known as the Wong–Zakai correction terms [13]. These terms can usually be split into two parts: one has the effect of modifying the conservative forces and another modifying the damping force. The first part can be combined with $-\partial H'/\partial Q_i$ to form overall effective conservative forces $-\partial H/\partial Q_i$ with the modified Hamiltonian $H = H(\mathbf{Q}, \mathbf{P})$ and with $\partial H/\partial P_i = \partial H'/\partial P_i$. The second part may be combined with $-\varepsilon c_{ij}\,\partial H'/\partial P_j$ to constitute effective damping forces $-\varepsilon m_{ij}\,\partial H/\partial P_j$ with $m_{ij} = m_{ij}(\mathbf{Q}, \mathbf{P})$. With these accomplished, equations (15a) and (15b) can be rewritten as

$$\mathrm{d}Q_i = \frac{\partial H}{\partial P_i}\,\mathrm{d}t, \tag{16a}$$

$$\mathrm{d}P_i = -\left(\frac{\partial H}{\partial Q_i} + \varepsilon m_{ij}\frac{\partial H}{\partial P_j}\right)\mathrm{d}t + \varepsilon^{1/2} f_{ik}\,\mathrm{d}B_k(t),$$

$$i, j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m. \tag{16b}$$

Assume that the Hamiltonian system with Hamiltonian H governed by equations (16a) and (16b) where $\varepsilon = 0$ is non-integrable, i.e., the Hamiltonian system has only one independent integral of motion, the Hamiltonian H. Then equations (16a) and (16b), also equations (13a) and (13b), describe a quasi-non-integrable-Hamiltonian system of n DOF.

It has been shown [10] that the response of the system governed by equations (16a) and (16b) converges in probability to a one-dimensional diffusion process of averaged Hamiltonian as $\varepsilon \to 0$. The averaged Hamiltonian is governed by the following *Itô* equation

$$\mathrm{d}H = \varepsilon U(H)\,\mathrm{d}t + \varepsilon^{1/2} V(H)\,\mathrm{d}B(t), \tag{17}$$

where

$$U(H) = \frac{1}{T(H)V_{\Omega_1}} \int_{\Omega} \left[\left(-m_{ij} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} + D_{kl} f_{ik} f_{jl} \frac{\partial^2 H}{\partial p_i \partial p_j} \right) \bigg/ \frac{\partial H}{\partial p_1} \right] \mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_2 \ldots \mathrm{d}p_n, \tag{18a}$$

$$V^2(H) = \frac{1}{T(H)V_{\Omega_1}} \int_{\Omega} \left[2D_{kl} f_{ik} f_{jl} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \bigg/ \frac{\partial H}{\partial p_1} \right] \mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_2 \ldots \mathrm{d}p_n, \tag{18b}$$

$$T(H) = \frac{1}{V_{\Omega_1}} \int_{\Omega} \left(1 \bigg/ \frac{\partial H}{\partial p_1} \right) \mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_2 \ldots \mathrm{d}p_n, \tag{18c}$$

$$V_{\Omega_1} = \int_{\Omega_1} \mathrm{d}q_2 \ldots \mathrm{d}q_n\, \mathrm{d}p_2 \ldots \mathrm{d}p_n, \tag{19}$$

in which domain Ω of the $(2n-1)$-fold integrals in equations (18a)–(18c) and domain Ω_1 of the $(2n-2)$-fold integral in equation (19) are defined as follows:

$$\Omega = \{(q_1, \ldots, q_n, p_2, \ldots, p_n) | H(q_1, \ldots, q_n, 0, p_2, \ldots, p_n) \leqslant H\}, \tag{20}$$

$$\Omega_1 = \{(q_2, \ldots, q_n, p_2, \ldots, p_n) | H(0, q_2, \ldots, q_n, 0, p_2, \ldots, p_n) \leqslant H\}. \tag{21}$$

4. ASYMPTOTIC STABILITY IN PROBABILITY OF QUASI-NON-INTEGRABLE-HAMILTONIAN SYSTEMS

The stability in probability and asymptotic stability in probability of the trivial solution of a system response described by a n-dimensional stochastic vector $\mathbf{x}(t)$ are defined as follows [2].

Stability in probability. The trivial solution is said to be stable in probability if, for every pair of $\varepsilon_1, \varepsilon_2 > 0$, there exists a $\delta(\varepsilon_1, \varepsilon_2, t_0) > 0$ such that

$$\text{prob}\,[\|\mathbf{X}(t; x_0, t_0)\| \geqslant \varepsilon_1] \leqslant \varepsilon_2 \qquad t \geqslant t_0, \tag{22}$$

provided $\|\mathbf{X}_0\| \leqslant \delta$, where $\mathbf{X}_0 = \mathbf{X}(t_0)$ is deterministic.

Asymptotic stability in probability. The trivial solution is said to be asymptotically stable in probability if equation (22) holds, and if for every $\varepsilon > 0$ exists a $\delta'(\varepsilon, t_0) > 0$ such that

$$\lim_{t\to\infty} \text{prob}\,[\|\mathbf{X}(t; x_0, t_0)\| \geqslant \varepsilon] = 0, \tag{23}$$

provided $\|\mathbf{X}_0\| \leqslant \delta'$.

In the above definition, the boundedness and convergence of $\mathbf{X}(t)$ are defined rigorously in terms of a suitable norm of $\mathbf{X}(t)$, denoted by $\|\mathbf{X}(t)\|$, for example,

$$\|\mathbf{X}(t)\| = \left[\sum_{i,j=1}^{n} a_{ij} X_i X_j \right]^{1/2}, \tag{24}$$

where a_{ij} are the elements of a positive definite square matrix.

In the last section it has been shown that the response of a quasi-non-integrable-Hamiltonian system converges in probability to a one-dimensional diffusion process of averaged Hamiltonian as $\varepsilon \to 0$. For a linear non-gyroscopic Hamiltonian system of n DOF, the Hamiltonian is of the form

$$H = \frac{1}{2} \sum_{i,j=1}^{n} (B_{ij} P_i P_j + C_{ij} Q_i Q_j), \tag{25}$$

where B_{ij} and C_{ij} are constants representing the system parameters. It is seen from the comparison of equations (24) and (25) that, for a quasi-non-integrable-Hamiltonian system, it is suitable to take $H^{1/2}$ as the norm of the response. This is also the reason why we define the Lyapunov exponent in terms of $H^{1/2}$ in examining stochastic stability of quasi-integrable-Hamiltonian systems [14]. Although there is slight inconsistency between norm $H^{1/2}$ and Euclidean norm, such as equation (24) for the non-linear Hamiltonian system, it is meaningful physically to take $H^{1/2}$ as the norm of the response since H is usually the total energy of the system. To define $H^{1/2}$ as a norm also simplifies the decision of the stochastic stability of quasi-Hamiltonian systems. Thus, instead of averaged Hamiltonian, we will examine the square-root of the averaged Hamiltonian, which is also a one-dimensional diffusion process.

Let

$$Y(t) = H^{1/2}(t). \tag{26}$$

The *Itô* stochastic differential equation governing $Y(t)$ is obtained from equation (17) by using the *Itô* differential rule as follows

$$\mathrm{d}Y = \varepsilon a(Y)\,\mathrm{d}t + (\varepsilon b(Y))^{1/2}\,\mathrm{d}B(t), \tag{27}$$

where

$$a(Y) = \tfrac{1}{2} Y^{-1} U(Y) - \tfrac{1}{8} Y^{-3} V^2(Y), \qquad b(Y) = \tfrac{1}{4} Y^{-2} V^2(Y), \tag{28}$$

in which $U(Y)$ and $V^2(Y)$ are obtained from $U(H)$ and $V^2(H)$ by replacing H with Y^2. For the one-dimensional diffusion process $Y(t)$ governed by *Itô* equation (27), the left boundary is at the trivial solution $Y = 0$ while the right boundary is usually at infinity, $H = \infty$, if H varies monotonously from 0 to ∞ and if there is no constraint imposed on the response of the system. For quasi-non-integrable-Hamiltonian systems with parametric excitations of Gaussian white noises, these boundaries are often singular.

Stability in probability implies that most sample functions of the response process remain near the trivial solution on the entire semi-infinite time domain

$t_0 \leqslant t < \infty$ provided they are near the trivial solution initially. Asymptotic stability in probability means that except the boundedness of most sample functions on $t_0 \leqslant t < \infty$, almost all sample functions of the response process converge to the trivial solution as time goes to infinity. For one-dimensional diffusion process $Y(t)$, it is obvious that the requirements of asymptotic stability in probability can be satisfied only if the trivial boundary is an exit or attractively natural, while the infinite boundary is an entrance or repulsively natural.

The necessary and sufficient conditions of the asymptotic stability in probability of the trivial solution of a quasi-non-integrable-Hamiltonian system governed by equations (13a) and (13b) are thus obtained approximately from examining the sample behaviors of one-dimensional diffusion process $Y(t)$ governed by equation (27) at boundaries $Y = 0, \infty$. To do so, asymptotic analysis is necessary for classifying the boundaries since $a(Y)$ and $b(Y)$ are defined in terms of multi-fold integrals. These will be shown in detail in the following example.

5. EXAMPLE

To illustrate the proposed procedure for the decision of asymptotic stability in probability of the trivial solution of the quasi-non-integrable-Hamiltonian systems by using the stochastic averaging method and by examining the sample behaviors of the one-dimensional diffusion process of the square-root of averaged Hamiltonian at the two boundaries, consider a system of linearly and non-linearly coupled two linearly and non-linearly damped oscillators subject to parametric excitations of Gaussian white noises. The equations of motion are of the form

$$\ddot{X} + \beta_1 \dot{X} + \alpha_1 X^2 \dot{X} + \omega_1^2 X + aY + b|X - Y|^{\delta} \operatorname{sign}(X - Y) = C_1 X W_1(t),$$

$$\ddot{Y} + \beta_2 \dot{Y} + \alpha_2 Y^2 \dot{Y} + \omega_2^2 Y + aX + b|X - Y|^{\delta} \operatorname{sign}(Y - X) = C_2 Y W_2(t), \quad (29)$$

where ω_1 and ω_2 are the natural frequencies of the two uncoupled oscillators; β_i and α_i are coefficients of linear and non-linear dampings; a and b are the constants of linear and non-linear couplings; δ is the power of non-linear coupling; C_i represent the amplitudes of excitations; α_i, β_i and C_i^2 are assumed of order ε; and $W_i(t)$ are independent Gaussian white noises with intensities $2D_i$. In the case of $\beta_i < 0$ and $\delta = 3$, the system has been studied by To and Lin [15] using the Stratonovich stochastic averaging method to determine the stochastic bifurcation, by Zhu *et al*. [16] using the equivalent non-linear system method for stochastically excited and dissipated non-integrable Hamiltonian systems, and by Zhu and Yang [10] using the stochastic averaging method for quasi-non-integrable-Hamiltonian systems to predict the response. Here we are going to obtain the necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution of the system.

It is noted that there is no Wong–Zakai correction term in this system. The Hamiltonian of the system is the total energy, i.e.

$$H = \tfrac{1}{2}(\dot{X}^2 + \dot{Y}^2) + U(X, Y), \quad (30)$$

where

$$U(X, Y) = \tfrac{1}{2}(\omega_1^2 X^2 + \omega_2^2 Y^2) + aXY + \frac{b}{1+\delta}|X - Y|^{1+\delta}. \tag{31}$$

The Hamiltonian system governed by equation (29) without damping and random excitations is usually non-integrable since $U(X, Y)$ is non-separable when $b \neq 0$ and $\delta \neq 0, 1$. The dampings are light and random excitations are weak. So equation (29) describes a quasi-non-integrable-Hamiltonian system.

Using the stochastic averaging method for the quasi-non-integrable-Hamiltonian systems introduced in section 3, one obtains the *Itô* equation for the averaged Hamiltonian

$$\mathrm{d}H = U(H)\,\mathrm{d}t + V(H)\,\mathrm{d}B(t), \tag{32}$$

where, according to equations (18a)–(19), the drift and diffusion coefficients are

$$U(H) = \frac{1}{T(H)V_{\Omega_1}} \int_\Omega \{[-(\beta_1 + \alpha_1 x^2)\dot{x}^2 - (\beta_2 + \alpha_2 y^2)\dot{y}^2 + C_1^2 D_1 x^2 + C_2^2 D_2 y^2]/\dot{x}\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\dot{y}, \tag{33a}$$

$$V^2(H) = \frac{1}{T(H)V_{\Omega_1}} \int_\Omega [2(C_1^2 D_1 x^2 \dot{x}^2 + C_2^2 D_2 y^2 \dot{y}^2)/\dot{x}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\dot{y}, \tag{33b}$$

$$T(H)V_{\Omega_1} = \int_\Omega \left(\frac{1}{\dot{x}}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\dot{y}, \tag{33c}$$

$$V_{\Omega_1} = \int_\Omega \mathrm{d}y\,\mathrm{d}\dot{y}. \tag{34}$$

and domains Ω and Ω_1 are defined as follows:

$$\Omega = \{(x, y, \dot{y}) | H(x, y, 0, \dot{y}) \leqslant H\}, \qquad \Omega_1 = \{(y, \dot{y}) | H(0, y, 0, \dot{y}) \leqslant H\}. \tag{35, 36}$$

Completing the integration of equations (33a)–(34) with respect to $\dot{y}$ and introducing co-ordinate transformations

$$x = \frac{R}{\omega_1}\cos\theta, \qquad y = \frac{R}{\omega_2}\sin\theta. \tag{37}$$

Equations (33a)–(34) become

$$U(H) = \frac{2\pi}{T(H)V_{\Omega_1}} \int_0^\pi \left[-(\beta_1 + \beta_2)A(H, \theta) - \left(\frac{\alpha_1}{\omega_1^2}\cos^2\theta + \frac{\alpha_2}{\omega_2^2}\sin^2\theta\right)B(H, \theta) + \frac{R^4}{2}\left(\frac{C_1^2 D_1}{\omega_1^2}\cos^2\theta + \frac{C_2^2 D_2}{\omega_2^2}\sin^2\theta\right)\right]\mathrm{d}\theta, \tag{38a}$$

$$V^2(H) = \frac{4\pi}{T(H)V_{\Omega_1}} \int_0^{\pi} \left[\left(\frac{C_1^2 D_1}{\omega_1^2} \cos^2\theta + \frac{C_2^2 D_2}{\omega_2^2} \sin^2\theta \right) B(H,\theta) \right] \mathrm{d}\theta, \qquad (38b)$$

$$T(H)V_{\Omega_1} = 2\pi \int_0^{\pi} R^2 \, \mathrm{d}\theta, \qquad (38c)$$

$$V_{\Omega_1} = \int_0^{\pi} r^2 \, \mathrm{d}\theta, \qquad (39)$$

where

$$A(H,\theta) = HR^2 - \frac{R^4}{4}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{2b}{(1+\delta)(3+\delta)} R^{3+\delta} \left| \frac{\cos\theta}{\omega_1} - \frac{\sin\theta}{\omega_2} \right|^{1+\delta}, \qquad (40)$$

$$B(H,\theta) = \frac{HR^4}{2} - \frac{R^6}{6}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{2b}{(1+\delta)(5+\delta)} R^{5+\delta} \left| \frac{\cos\theta}{\omega_1} - \frac{\sin\theta}{\omega_2} \right|^{1+\delta}, \qquad (41)$$

and R and r are the solutions of the following two equations:

$$H - \frac{R^2}{2}\left(1 + \frac{a}{\omega_1\omega_2}\sin 2\theta\right) - \frac{b}{(1+\delta)} R^{1+\delta} \left| \frac{\cos\theta}{\omega_1} - \frac{\sin\theta}{\omega_2} \right|^{1+\delta} = 0, \qquad (42)$$

$$H - \frac{r^2}{2}\sin^2\theta - \frac{r^2}{2}\omega_2^2\cos^2\theta - \frac{b}{(1+\delta)} r^{1+\delta} |\cos\theta|^{1+\delta} = 0. \qquad (43)$$

The drift and diffusion coefficients of the one-dimensional diffusion process of the square-root of averaged Hamiltonian, $Y = H^{1/2}$, are then

$$a(Y) = \tfrac{1}{2} Y^{-1} U(Y) - \tfrac{1}{8} Y^{-3} V^2(Y), \qquad b(Y) = \tfrac{1}{4} Y^{-2} V^2(Y), \qquad (44, 45)$$

where $U(Y)$ and $V^2(Y)$ are obtained from equations (38a) and (38b) with H replaced by Y^2. In the following, the classification of the boundaries $Y = 0, \infty$ of the one-dimensional diffusion process Y will be identified and the necessary and sufficient conditions will be approximately obtained for the asymptotic stability in probability of the trivial solution of system (29) through asymptotic analysis of $a(Y)$ and $b(Y)$ near the boundaries $Y = 0, \infty$. Four cases are examined with the emphasis placed on the effects of non-linear damping and non-linear coupling on the stability conditions.

Case 1: $\alpha_i \neq 0$ and $0 < \delta < 1$. First consider the left boundary $Y = 0$. It is seen from equation (42) that $R \to 0$ as $H \to 0$ and the third term is dominant compared with the second term on the left side of equation (42) except that θ is in the interval $\theta \in [\theta_0 - \theta_1, \theta_0 + \theta_1]$ where $|(\cos\theta/\omega_1) - (\sin\theta/\omega_2)|^{1+\delta}$ is very small. Thus, as $H \to 0$,

$$R^2 \to \frac{\left(2 - \frac{|\theta - \theta_0|}{\theta_1}\right) H}{1 + \frac{a}{\omega_1 \omega_2} \sin 2\theta}, \qquad \theta \in [\theta_0 - \theta_1, \theta_0 + \theta_1], \tag{46}$$

$$R \to \left[\frac{(1+\delta)\beta H}{b}\right]^{1/(1+\delta)} \bigg/ \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_1 \omega_2} |\sin(\theta - \theta_0)|,$$

$$\theta \in [0, \theta_0 - \theta_1) \quad \text{and} \quad \theta \in (\theta_0 + \theta_1, \pi], \tag{47}$$

where

$$\theta_0 = \operatorname{arctg} \frac{\omega_2}{\omega_1}$$

$$\theta_1 = \left(\frac{1+\delta}{b}\right)^{1/(1+\delta)} \frac{\omega_1 \omega_2}{\sqrt{\omega_1^2 + \omega_2^2}} \left(\frac{1}{2} + \frac{a}{\omega_1^2 + \omega_2^2}\right)^{1/2} H^{(1-\delta)/2(1+\delta)},$$

$$\beta = 1 \quad \text{for} \quad \theta \leqslant \theta_0 - 0{\cdot}05\pi \quad \text{or} \quad \theta \geqslant \theta_0 + 0{\cdot}05\pi,$$

$$\beta = 1 - \frac{\theta - \theta_0 + 0{\cdot}05\pi}{2(0{\cdot}05\pi - \theta_1)} \quad \text{for} \quad (\theta_0 - 0{\cdot}05\pi) < \theta < (\theta_0 - \theta_1),$$

$$\beta = 1 + \frac{\theta - \theta_0 - 0{\cdot}05\pi}{2(0{\cdot}05\pi - \theta_1)} \quad \text{for} \quad (\theta_0 + 0{\cdot}05\pi) > \theta > (\theta_0 + \theta_1). \tag{48}$$

The exact solution of equation (42) and the approximate solution in equations (46) and (47) are compared in Appendix A. It is seen that they are in good agreement. Substituting equations (46) and (47) into equations (40) and (41) and then into equations (38a)–(38c) and completing the integration in equations (38a)–(39) (see Appendix B) leads to

$$U(H) = \lambda_1 H + o(H) \quad \text{as} \quad H \to 0, \tag{49}$$

$$V^2(H) = \lambda_2 H^2 + o(H^2) \quad \text{as} \quad H \to 0, \tag{50}$$

where $o(H)$ denotes a term one-order smaller than H, λ_1 and λ_2 are

$$\lambda_1 = -\frac{1}{3}(\beta_1 + \beta_2)\eta_1 + \frac{7}{9}\frac{C_1^2 D_1 + C_2^2 D_2}{2a + \omega_1^2 + \omega_2^2}, \qquad \lambda_2 = \frac{2}{3}\frac{(C_1^2 D_1 + C_2^2 D_2)}{(2a + \omega_1^2 + \omega_2^2)}\eta_2, \tag{51, 52}$$

in which

$$\eta_1 = \frac{11}{6} - \frac{1+\delta}{4(4+\delta)} - \frac{2(2-\delta)}{(2+\delta)(3+\delta)} + \frac{\delta^2 - 1}{48(5+\delta)},$$

$$\eta_2 = \frac{13}{12} - \frac{16}{(5+\delta)(2+\delta)} + \frac{4}{3+\delta} - \frac{3+\delta}{2(4+\delta)} + \frac{(3+\delta)(1+\delta)}{24(5+\delta)}.$$

The asymptotic expressions for the drift and diffusion coefficients of $Y(t)$ at $Y=0$ are thus

$$a(Y)=\tfrac{1}{8}(4\lambda_1-\lambda_2)Y+o(Y),\quad \text{as}\quad Y\to 0, \tag{53}$$

$$b(Y)=\tfrac{1}{4}\lambda_2 Y^2+o(Y^2),\quad \text{as}\quad Y\to 0. \tag{54}$$

The left boundary $Y=0$ is a singular one of the first kind and according to equations (4)–(6), the diffusion exponent, drift exponent and character value are

$$\alpha_l=2,\quad \beta_l=1,\quad c_l=\frac{4\lambda_l-\lambda_2}{\lambda_2}. \tag{55}$$

It is seen from Table 2 that the further classification of the boundary depends on the value of c_l: it is

attractively natural if $c_l<1$,

strictly natural if $c_l=1$,

repulsively natural if $c_l>1$. (56)

Now consider the right boundary $Y=\infty$. It is seen from equation (42) that the second term is dominant compared with the third term as $H\to\infty$. Thus,

$$R^2\to\frac{2H}{1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta},\quad \text{as}\quad H\to\infty. \tag{57}$$

Substituting equation (57) into equations (40) and (41) and then into equations (38a)–(38c) and completing the integration in equations (38a)–(39) leads to

$$U(H)=-\frac{1}{6}\left(\frac{\alpha_1}{\omega_1^2}+\frac{\alpha_2}{\omega_2^2}\right)\eta H^2+o(H^2), \tag{58}$$

$$V^2(H)=\frac{1}{3}\left(\frac{C_1^2D_1}{\omega_1^2}+\frac{C_2^2D_2}{\omega_2^2}\right)\eta H^2+o(H^2), \tag{59}$$

where

$$\eta=\int_0^\pi\frac{\mathrm{d}\theta}{\left(1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta\right)^2}\Bigg/\int_0^\pi\frac{\mathrm{d}\theta}{1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta}. \tag{60}$$

The drift and diffusion coefficients of the one-dimensional diffusion process $Y(t)$ are thus

$$a(Y)=-\frac{1}{12}\left(\frac{\alpha_1}{\omega_1^2}+\frac{\alpha_2}{\omega_2^2}\right)\eta Y^3+o(Y^3), \tag{61}$$

$$b(Y) = \frac{1}{12}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\eta Y^2 + o(Y^2). \tag{62}$$

The right boundary $Y = \infty$ is a singular boundary of the second kind. The diffusion exponent, drift exponent and character value at the right boundary, according to equations (10)–(12), are

$$\alpha_r = 2, \quad \beta_r = 3, \quad c_r = \frac{2(\alpha_1\omega_2^2 + \alpha_2\omega_1^2)}{C_1^2 D_1 \omega_2^2 + C_2^2 D_2 \omega_1^2}. \tag{63}$$

It is seen from Table 4 that the right boundary is always the entrance provided that $(\alpha_1\omega_2^2 + \alpha_2\omega_1^2) > 0$.

The behaviors of the two boundaries are schematically shown in Figure 1. The only case where $Y = 0$ is asymptotic stable in probability is when the left boundary is attractively natural while the right boundary is the entrance. Therefore, the necessary and sufficient conditions for the asymptotic stability in probability of the trivial solution of system (29) in this case are approximately

$$(\alpha_1\omega_2^2 + \alpha_2\omega_1^2) > 0 \tag{64}$$

and

$$\beta_1 + \beta_2 > \frac{(C_1^2 D_1 + C_2^2 D_2)}{(2a + \omega_1^2 + \omega_2^2)}\left[\frac{7}{3\eta_1} - \frac{\eta_2}{\eta_1}\right]. \tag{65}$$

If equations (64) and (65) are satisfied, the stationary probability density of the system response is a *delta* function at the trivial solution. In the case of $c_l > 1$, a non-*delta* type stationary probability density exists since both the two boundaries are now unreachable if a sample path begins from an interior point. The existence of a non-*delta* type stationary probability density implies that the trivial solution is unstable in probability.

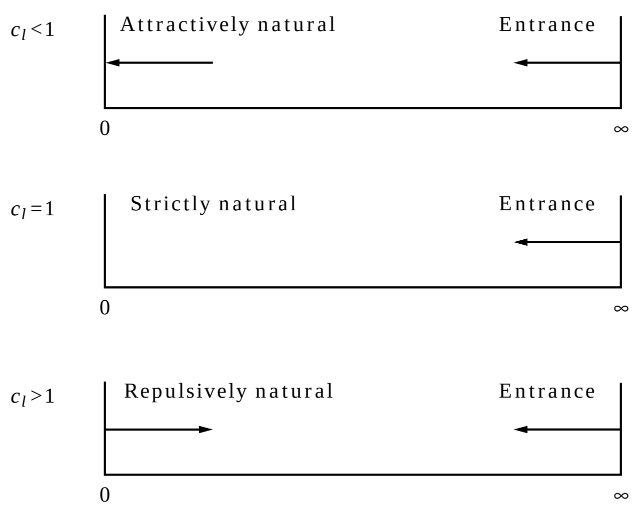

Figure 1. The behavior of the two boundaries.

Case 2: $\alpha_i = 0$ and $0 < \delta < 1$. In this case the behavior of the left boundary is the same as in Case 1. It can be shown that at the right boundary

$$a(Y) = \frac{1}{4}\left[-(\beta_1 + \beta_2) + \frac{5\eta}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\right] Y + o(Y), \tag{66}$$

$$b(Y) = \frac{\eta}{12}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\Bigg] Y^2 + o(Y^2). \tag{67}$$

The right boundary is a singular one of the second kind. The diffusion exponent, drift exponent and character value, according to equations (10)–(12), are

$$\alpha_r = 2, \quad \beta_r = 1, \quad c_r = \frac{-\left[-(\beta_1 + \beta_2) + \frac{5\eta}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\right]}{\left[\frac{\eta}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\right]}. \tag{68}$$

The right boundary is repulsively natural, strictly natural or attractively natural depending on $c_r > -1$, $c_r = -1$ or $c_r < -1$, respectively. There are in total nine possible combinations of the behaviors of the two boundaries and the only case where the trivial solution is asymptotically stable in probability is when the left boundary is attractively natural while the right boundary is repulsively natural. Therefore, the necessary and sufficient condition for the asymptotic stability in probability of the trivial solution of system (29) is approximately equation (65), or

$$\beta_1 + \beta_2 > \frac{2}{3}\eta\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right), \tag{69}$$

depending on which is more severe.

Case 3: $\alpha_i \neq 0$ and $\delta > 1$. In this case the second term in equation (42) is dominant compared with the third term for small H. Thus,

$$R^2 \to \frac{2H}{1 + \frac{a}{\omega_1 \omega_2}\sin 2\theta}, \quad \text{as} \quad H \to 0. \tag{70}$$

Substituting equation (70) into equations (40) and (41), then into equations (38a)–(38c) and completing the integration in equations (38a)–(39), one obtains

$$a(Y) = \frac{1}{4}\left[-(\beta_1 + \beta_2) + \frac{5}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\eta \right] Y + o(Y), \tag{71}$$

$$b(Y) = \frac{1}{12}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\Bigg]\eta Y^2 + o(Y^2). \tag{72}$$

The left boundary is a singular one of the first kind. The diffusion exponent, drift exponent and character value, defined by equations (4)–(6), are

$$\alpha_l = 2, \quad \beta_l = 1, \quad c_l = \frac{\left[-(\beta_1+\beta_2) + \frac{5}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\eta\right]}{\left[\frac{1}{6}\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right)\eta\right]}. \tag{73}$$

According to Table 2, the left boundary $Y = 0$ is attractively natural, strictly natural or repulsively natural depending on whether $c_l < 1$, $c_l = 1$ or $c_l > 1$, respectively.

In equation (42), the third term is dominant compared with the second term in the intervals $\theta \in [0, \theta_0 - \theta_1)$ and $\theta \in (\theta_0 + \theta_1, \pi]$ while the second term is dominant in the interval $\theta \in [\theta_0 - \theta_1, \theta_0 + \theta_1]$ for $H \to \infty$. Thus, equations (46) and (47) hold in this case. Using the same procedure as that leading to equations (53) and (54), one obtains

$$a(Y) = -\frac{\eta_2}{6}\frac{\alpha_1 + \alpha_2}{2a + \omega_1^2 + \omega_2^2}Y^3 + o(Y^3), \tag{74}$$

$$b(Y) = \frac{\eta_2}{6}\frac{C_1^2 D_1 + C_2^2 D_2}{2a + \omega_1^2 + \omega_2^2}Y^2 + o(Y^2). \tag{75}$$

The right boundary $Y = \infty$ is a singular one of the second kind. The diffusion exponent, drift exponent and character value, defined in equations (10)–(12), are

$$\alpha_r = 2, \quad \beta_r = 3, \quad c_r = \frac{2(\alpha_1 + \alpha_2)}{(C_1^2 D_1 + C_2^2 D_2)} \tag{76}$$

and $a(\infty) < 0$.

According to Table 4, the right boundary is an entrance. The trivial solution $Y = 0$ is asymptotically stable in probability when the left boundary is attractively natural. Therefore, the necessary and sufficient condition for the asymptotic stability in probability of the trivial solution of system (29) is approximately

$$\beta_1 + \beta_2 > \frac{2}{3}\eta\left(\frac{C_1^2 D_1}{\omega_1^2} + \frac{C_2^2 D_2}{\omega_2^2}\right). \tag{77}$$

Case 4: $\alpha_i = 0$ and $\delta > 1$. In this case the left boundary is the same as in Case 3. As for the right boundary, it can be shown that

$$a(Y) = \left[-\frac{\beta_1 + \beta_2}{6}\eta_1 + \frac{(C_1^2 D_1 + C_2^2 D_2)}{2a + \omega_1^2 + \omega_2^2}\left(\frac{7}{18} - \frac{\eta_2}{12}\right)\right]Y + o(Y), \tag{78}$$

$$b(Y) = \frac{\eta_2}{6}\frac{C_1^2 D_1 + C_2^2 D_2}{2a + \omega_1^2 + \omega_2^2}Y^2 + o(Y). \tag{79}$$

The right boundary is a singular one of the second kind. The diffusion exponent, drift exponent and character value, according to equations (10)–(12), are

$$\alpha_r = 2, \quad \beta_r = 1, \quad c_r = \frac{2(2a + \omega_1^2 + \omega_2^2)(\beta_1 + \beta_2)\eta_1}{(C_1^2 D_1 + C_2^2 D_2)\eta_2} + \left(1 - \frac{14}{3\eta_2}\right). \tag{80}$$

It is seen from Table 4 that the right boundary is attractively natural, strictly natural or repulsively natural depending for $c_r < -1$, $c_r = -1$ or $c_r > -1$, respectively. The trivial solution $Y = 0$ is asymptotic stable in probability when the left boundary is attractively natural while the right boundary is repulsively natural. Thus, the necessary and sufficient condition for the asymptotic stability in probability of the trivial solution of system (29) is equation (77) or

$$\beta_1 + \beta_2 > \left(\frac{7}{3\eta_1} - \frac{\eta_2}{\eta_1}\right)\frac{C_1^2 D_1 + C_2^2 D_2}{2a + \omega_1^2 + \omega_2^2}, \tag{81}$$

depending on which is more severe.

6. CONCLUSION

In the present paper a procedure for obtaining approximately the necessary and sufficient conditions for asymptotic stability in probability of the trivial solution of quasi-non-integrable-Hamiltonian systems has been developed. It has been suggested that the asymptotic stability in probability for this kind of systems is defined in terms of the square-root of the Hamiltonian. The stochastic averaging method for quasi-non-integrable-Hamiltonian systems has been employed to reduce such a system to a one-dimensional diffusion process of averaged Hamiltonian. Then the *Itô* stochastic differential equation governing the one-dimensional diffusion process of the square-root of averaged Hamiltonian was obtained by using the *Itô* differential rule. The asymptotic stability in probability of the trivial solution of the quasi-non-integrable-Hamiltonian systems was determined by examining the sample behaviors of the one-dimensional diffusion process of the square-root of averaged Hamiltonian at the two boundaries. A system of linearly and non-linearly coupled two non-linearly damped oscillators subject to parametric excitations of Gaussian white noises was used as an example to illustrate the proposed procedure, and the effects of non-linear damping and non-linear coupling on the stability conditions were analysed in detail.

ACKNOWLEDGMENT

The work reported in this paper is supported by the National Natural Science Foundation of China under grant no. 19672054 and the Special Fund for Doctor Programs in Institutions of Higher Learning of China and the Cao Guang Biao Science Foundation of Zhejiang University.

REFERENCES

1. F. Kozin 1969 *Automatica* **5**, 95–112. A survey of stability of stochastic systems.
2. Y. K. Lin and G. Q. Cai 1995 *Probabilistic Structural Dynamics, Advanced Theory and Applications*. New York: McGraw-Hill, Inc.
3. F. Kozin and Y. Sunahara 1987 *Proceedings of the 2nd Midwestern Mechanics Conference, West Lafayette, IN* **14**(a), 291–298. An application of the averaging method to noise stabilization of nonlinear system.
4. N. Sri Namachchivaya 1989 *Solid Mechanics Archives* **14**, 131–142. Instability theorem based on the nature of the boundary behaviour for one-dimensional diffusion.
5. Z. Y. Zhang 1991 *Ph.D dissertation, Polytechnic University, New York*. New developments in almost-sure sample stability of nonlinear stochastic dynamical systems.
6. Y. K. Lin and G. Q. Cai 1994 *International Journal of Nonlinear Mechanics* **29**, 539–553. Stochastic stability of nonlinear systems.
7. R. L. Stratonovich 1963 *Topics in the Theory of Random Noise*, Volume 1. New York: Gordon and Breach.
8. P. S. Landa and R. L. Stratonovich 1962 *Vestinik MGU* (Proc. of Moscow University), Series III(1), 33–45. Theory of stochastic transitions of various systems between different states (in Russian).
9. R. Z. Khasminskii 1964 *Prikladnaya Matematika i Mechanica* (Applied Mathematics and Mechanics) **28**, 1126–1130. On the behavior of a conservative system with small friction and small random noise.
10. W. Q. Zhu and Y. Q. Yang 1997 *ASME Journal of Applied Mechanics* **64**, 157–164. Stochastic averaging of quasi-nonintegrable-Hamiltonian systems.
11. W. Feller 1952 *Annals of Mathematics* **55**, 468–519. The parabolic differential equation and the associated semigroups of transformations.
12. S. Karlin and H. M. Taylor 1981 *A Second Course in Stochastic Process*. New York: Academic Press.
13. E. Wong and M. Zakai 1965 *International Journal of Engineering Science* **3**, 213–229. On the relation between ordinary and stochastic equations.
14. W. Q. Zhu and Z. L. Huang 1998 (in press) *ASME Journal of Applied Mechanics*. Lyapunov exponents and stochastic stability of quasi-integrable-Hamiltonian systems.
15. C. W. S. To and R. Lin 1989 *Structural Safety* **6**, 223–231. Bifurcation in a stochastically disturbed nonlinear two-degree-of-freedom system.
16. W. Q. Zhu, T. T. Soong and Y. Lei 1994 *ASME Journal of Applied Mechanics* **61**, 618–623. Equivalent nonlinear system method for stochastically excited Hamiltonian systems.
17. R. Z. Khasminskii 1967 *Theory of Probability and Application* **12**, 144–147. Sufficient and necessary conditions of almost-sure asymptotic stability of a linear stochastic system.

APPENDIX A

Let $H = 10^{-6}$, $a = \delta = 0{\cdot}5$, $\omega_1 = \omega_2 = b = 1$. One obtains from equation (48) that $\theta_0 = \pi/4$ and $\theta_1 = 0{\cdot}08024$. The exact solution of equation (42) and the approximate solution obtained from equations (46) and (47) are compared in Figure 2.

APPENDIX B

The integral in equation (38a) can be divided into three parts

$$\int_0^{\pi} [\,\cdot\,]\,\mathrm{d}\theta = \int_0^{\theta_0-\theta_1} [\,\cdot\,]\,\mathrm{d}\theta + \int_{\theta_0-\theta_1}^{\theta_0+\theta_1} [\,\cdot\,]\,\mathrm{d}\theta + \int_{\theta_0+\theta_1}^{\pi} [\,\cdot\,]\,\mathrm{d}\theta, \tag{B1}$$

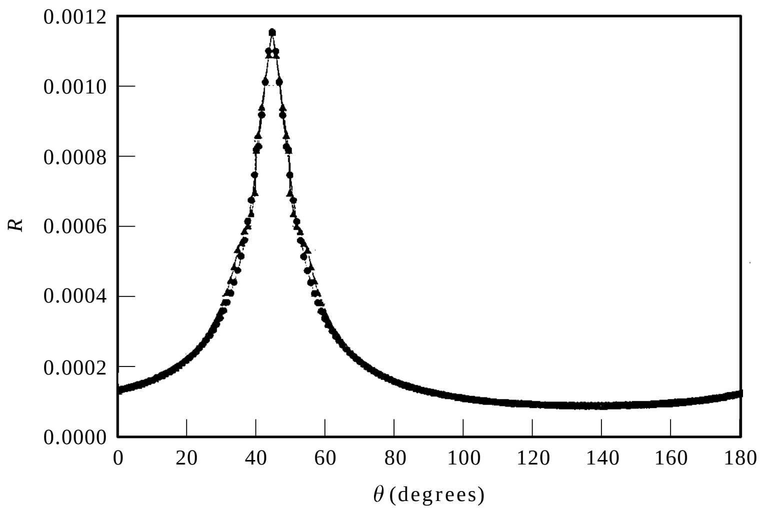

Figure 2. The exact solution of equation (42) and the approximate solution obtained from equations (46) and (47). —▲—, Approximate solution; —●—, exact solution.

where [·] represents the integrand in the integrals. R in the integrand of the second integral on the right side of equation (B1) takes the value in equation (46) while those in the first and third integrals the value in equation (47). The second integral is dominant since it is proportional to $H^{1+(3+\delta)/2(1+\delta)}$ while the first and the third ones to $H^{1+(4/2(1+\delta))}$ as $H \to 0$. Also, $\theta_1 \to 0$ as $H \to 0$, so $\cos^2\theta$, $\sin^2\theta$ and $\sin 2\theta$ in the integrand can be replaced approximately by $\cos^2\theta_0$, $\sin^2\theta_0$ and $\sin 2\theta_0$, respectively, and $\sin(\theta-\theta_0)$ by $\theta-\theta_0$ since $0 \leqslant \theta-\theta_0 \leqslant \theta_1$. Thus,

$$\begin{aligned}\int_0^{\pi}[\cdot]\,\mathrm{d}\theta &\doteq \int_{\theta_0-\theta_1}^{\theta_0+\theta_1}[\cdot]\,\mathrm{d}\theta = 2\int_{\theta_0}^{\theta_0+\theta_1}[\cdot]\,\mathrm{d}\theta \doteq 2(\alpha_1+\alpha_2)\int_{\theta_0}^{\theta_0+\theta_1} A(H,\theta)\,\mathrm{d}\theta \\ &\quad - 2\left(\frac{\beta_1}{\omega_1^2}\cos^2\theta_0 + \frac{\beta_2}{\omega_2^2}\sin^2\theta_0\right)\int_{\theta_0}^{\theta_0+\theta_1} B(H,\theta)\,\mathrm{d}\theta \\ &\quad + \left(\frac{C_1^2 D_1}{\omega_1^2}\cos^2\theta_0 + \frac{C_2^2 D_2}{\omega_2^2}\sin^2\theta_0\right)\int_{\theta_0}^{\theta_0+\theta_1} R^4\,\mathrm{d}\theta, \qquad \text{(B2)}\end{aligned}$$

where

$$\begin{aligned}\int_{\theta_0}^{\theta_0+\theta_1} A(H,\theta)\,\mathrm{d}\theta &\doteq \frac{11H^2\theta_1}{12\left(1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta_0\right)} \\ &\quad - \frac{2b}{(1+\delta)(3+\delta)}\left(\frac{2H}{1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta_0}\right)^{(3+\delta)/2} \\ &\quad \times \left(\frac{\sqrt{\omega_1^2+\omega_2^2}}{\omega_1\omega_2}\right)^{1+\delta}\int_0^{\theta_1}\left(1-\frac{\theta}{2\theta_1}\right)^{(3+\delta)/2}\theta^{1+\delta}\,\mathrm{d}\theta, \qquad \text{(B3)}\end{aligned}$$

$$\int_{\theta_0}^{\theta_0+\theta_1} B(H,\theta)\,\mathrm{d}\theta \doteq \frac{13H^2\theta_1}{24\left(1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta_0\right)^2} - \frac{2b}{(1+\delta)(5+\delta)}\left(\frac{2H}{1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta_0}\right)^{(5+\delta)/2} \times \left(\frac{\sqrt{\omega_1^2+\omega_2^2}}{\omega_1\omega_2}\right)^{1+\delta}\int_0^{\theta_1}\left(1-\frac{\theta}{2\theta_1}\right)^{(5+\delta)/2}\theta^{1+\delta}\,\mathrm{d}\theta, \quad \text{(B4)}$$

$$\int_{\theta_0}^{\theta_0+\theta_1} R^4\,\mathrm{d}\theta \doteq \frac{7H^2\theta_1}{3\left(1+\dfrac{a}{\omega_1\omega_2}\sin 2\theta_0\right)}. \quad \text{(B5)}$$

To complete the integration in equations (B3) and (B4), the integrands are first expanded into power series in θ and then the terms higher than θ^3 are neglected. The results are

$$\int_0^{\theta_1}\left(1-\frac{\theta}{2\theta_1}\right)^{(3+\delta)/2}\theta^{1+\delta}\,\mathrm{d}\theta \doteq \left[\frac{1}{2+\delta}-\frac{1}{4}+\frac{(3+\delta)(1+\delta)}{32(4+\delta)} - \frac{(3+\delta)(\delta^2-1)}{384(5+\delta)}\right]\theta_1^{2+\delta}, \quad \text{(B6)}$$

$$\int_0^{\theta_1}\left(1-\frac{\theta}{2\theta_1}\right)^{(5+\delta)/2}\theta^{1+\delta}\,\mathrm{d}\theta \doteq \left[\frac{1}{2+\delta}-\frac{5+\delta}{4(3+\delta)}+\frac{(5+\delta)(3+\delta)}{32(4+\delta)} - \frac{(3+\delta)(1+\delta)}{384}\right]\theta_1^{2+\delta}. \quad \text{(B7)}$$

The relative errors in the integrals (B6) and (B7) caused by neglecting the higher order terms are less than 10^{-3}.

The integral in equation (38b) can be treated similarly.